Solution to Exercise 4

1. Find the interior points and boundary points of the following sets:

   (a) $E_1 = \{(x, y) : x \in [0, a], y \in [0, b]\}$.

   (b) $E_2 = \{(x, y, z) : z > x^2 + y^2 - 1\}$.

   (c) $E_3 = \{(x, y, z) : 1 < x^2 + y^2 + z^2 \leq 4\}$.

   (d) $E_4 = \{(x, y) : x \in [0, 1]\}$.

   (e) $E_5 = \{(x, y) : x, y \in \mathbb{Z}\}$.

   (f) $E_6 = \left\{(x, y) : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\right\}, \quad a, b > 0$.

   The determine whether these sets are open, closed, or compact. A set is compact if it is closed and bounded.

   Solution.

   (a) Interior points: $(0, a) \times (0, b)$;
   Boundary points: $\{0\} \times [0, b] \cup \{a\} \times [0, b] \cup [0, a] \times \{0\} \cup [0, a] \times \{b\}$

   (b) Interior points: E_2
   Boundary points: $\{(x, y, z) : z = x^2 + y^2 - 1\}$

   (c) Interior points: $\{(x, y, z) : 1 < x^2 + y^2 + z^2 < 4\}$
   Boundary points: $\{(x, y, z) : x^2 + y^2 + z^2 = 1\} \cup \{(x, y, z) : x^2 + y^2 + z^2 = 4\}$

   (d) Interior points: $\{(x, y) : x \in (0, 1)\}$
   Boundary points: $\{(x, y) : x = 0\} \cup \{(x, y) : x = 1\}$

   (e) Interior points: ϕ
   Boundary points: E_5

   (f) Interior points: ϕ
   Boundary points: E_6

	open	closed	compact
E_1	N	Y	Y
E_2	Y	N	N
E_3	N	N	N
E_4	N	Y	N
E_5	N	Y	N
E_6	N	Y	Y

2. Let A and B be open sets in $\mathbb{R}^n$. Show that

   (a) $A \bigcup B$ is open.

   (b) $A \bigcap B$ is open.

   Solution. (a) Let $x \in A \bigcup B$. Then $x \in A$ or $x \in B$. If $x \in A$, there is some $B(x) \subset A$ since A is open. But then $B(x) \subset A \bigcup B$. The same conclusion holds if $x \in B$. The desired result follows. (b) Let $x \in A \bigcap B$. As $x \in A$ and A is open, there is some $B_r(x) \subset A$. Similarly, there is some $B_\rho(x) \subset B$. Consequently, the ball $B_s(x)$, $s = \min\{r, \rho\}$, is contained in $A \bigcap B$, so $A \bigcap B$ is open.

Note. In fact, any union of open sets is still open and any intersection of closed sets is closed. On the other hand, any finite intersection of open sets is open and any finite union of closed sets is closed. But infinite intersection of open sets may not be open. For instance, each interval $(-n, n), \geq 1$, is open but $\bigcap_n(-n, n) = \{0\}$ is a singleton which is not open. Similarly, there are infinite unions of closed sets which are not closed.

3. (a) Show that $(a, b), -\infty \leq a < b \leq \infty$, is open.
   (b) Show that $[a, b], -\infty < a \leq b < \infty$, is open. Note that it implies that the singleton set $\{a\}$ is closed.

   Solution. (a) is obvious. (b) Observe that $\mathbb{R} \setminus [a, b] = (-\infty, a) \bigcup (b, \infty)$ is a union of two open sets.

4. * Prove that F is a closed set in $\mathbb{R}^n$ if and only if every convergent sequence in F has its limit in F.

   Solution. Let F be closed and $x_n \in F, x_n \to x$. We want to show that $x \in F$. If it is not, as F is closed, $\mathcal{C}F$ is open, there is some ball $B(x) \subset \mathcal{C}F$. But then $x_n \in F$ cannot converge to x, contradiction holds.

   Conversely, assuming that F is not closed, then $\mathcal{C}F$ is not open. That means there is some $x \in \mathcal{C}F$ such that $B_{1/n}(x) \bigcap F \neq \phi$ for all n. Pick a point x_n in this intersection to form a sequence $\{x_n\}$. Clearly, $x_n \in F$ but $x_n \to x \neq F$.

5. * Prove that whenever F is a closed set containing E, then it must also contain $\overline{E}$. It shows that the closure of a set is the smallest closed set containing this set.

   Solution. Let $x \in \partial E$. From the definition of the boundary point there is a sequence $x_n \in E, x_n \to x$. But x_n is also a sequence in F. As F is closed, the limit of x_n, that is x, must belong to F. We have shown that $\partial E \subset F$.

6. Study the limit of the following functions at $(0, 0)$.

   (a)
   $$f(x, y) = \frac{x^2y^2}{|x| + y^2} \ .$$

   (b)
   $$g(x, y) = \frac{\sin xy}{x^2 + y^2} \ .$$

   (c)
   $$h(x, y) = y \log(x^2 + |y|) \ .$$

   Hint: In (c) examine the sets $\{(x, y) : y \geq x^2\}$ and $\{(x, y) : \ y < x^2\}$ separately.

   Solution. (a) We have
   $$0 \leq f(x, y) = \frac{x^2y^2}{|x| + y^2} \leq \frac{x^2y^2}{y^2} = x^2 \ .$$

   By Sandwich Rule,
   $$0 \leq \lim_{(x,y)\to(0,0)} f(x, y) \leq \lim_{(x,y)\to(0,0)} x^2 = 0 \ ,$$

so

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \ .$$

(b) Along the sequence $(x_n, 0), x_n \downarrow 0, g(x_n, 0)| = 0$ hence

$$\lim_{(x_n,0)\to(0,0)} g(x,y) = 0 \ .$$

On the other hand, along $(x_n, x_n), x_n \downarrow 0$,

$$g(x_n, x_n) = \frac{\sin x_n^2}{2x_n^2} \to \frac{1}{2}, \quad n \to \infty \ .$$

So

$$\lim_{(x_n,x_n)\to(0,0)} g(x,y) = \frac{1}{2} \ .$$

There are two sequences converging to $(0,0)$ with different limits, so $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. (We have used the fact that $\sin t/t \to 1$ as $t \to 0$.)

(c) Indeed, we have

$$\log(|y|) \le \log(|y| + x^2) < 0,$$

when x, y is close enough to $(0,0)$. Then,

$$0 \le |h(x,y)| \le |y||\log(|y| + x^2)| \le |y||\log(|y|)| = |y \log(|y|)|.$$

By Sandwich Rule,

$$0 \le \lim_{(x,y)\to(0,0)} |h(x,y)| \le \lim_{(x,y)\to(0,0)} |y \log(|y|)| = 0,$$

so

$$\lim_{(x,y)\to(0,0)} h(x,y) = 0.$$

(We have used the fact that $t \log t \to 0$ as $t \downarrow 0$.)

7. Find the iterated limits and limit of the function

$$h(x,y) = \frac{x-y}{x+y}$$

at $(0,0)$.

Solution.

$$\lim_{y\to 0} \lim_{x\to 0} \frac{x-y}{x+y} = \lim_{y\to 0} -1 = -1 \ ,$$

and

$$\lim_{x\to 0} \lim_{y\to 0} \frac{x-y}{x+y} = \lim_{x\to 0} 1 = 1 \ .$$

They are not equal.

8. Consider the function

$$F(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} \ .$$

Show that

$$\lim_{y\to 0} \lim_{x\to 0} F(x,y) = \lim_{x\to 0} \lim_{y\to 0} F(x,y) = 0 \ ,$$

but

$$\lim_{(x,y)\to(0,0)} F(x,y)$$

does not exist.

Solution.

$$\lim_{y\to 0}\lim_{x\to 0} F(x,y) = \lim_{y\to 0}(0) = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} F(x,y) = \lim_{x\to 0}(0) = 0$$

Therefore,

$$\lim_{y\to 0}\lim_{x\to 0} F(x,y) = \lim_{x\to 0}\lim_{y\to 0} F(x,y) = 0 \ ,$$

We show $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist by constructing two sequences converging to $(0,0)$ but with different limit of F: Consider $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$. Then $(x_n, y_n) \to (0,0)$ as $n \to \infty$, and $F(x_n, y_n) = \frac{\frac{1}{n^4}}{\frac{1}{n^4}} = 1$, and hence

$$\lim_{n\to\infty} F(x_n, y_n) = 1$$

Consider $(x'_n, y'_n) = (\frac{1}{n}, \frac{2}{n})$. Then $(x_n, y_n) \to (0,0)$ as $n \to \infty$, and $F(x_n, y_n) = \frac{\frac{4}{n^4}}{\frac{5}{n^4}} = \frac{4}{5}$, and hence

$$\lim_{n\to\infty} F(x'_n, y'_n) = \frac{4}{5}$$

Therefore, $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist.

9. Describe the natural domains of the functions determined by the following formulas and then study the continuity of these functions.

   (a) $\dfrac{1}{x^2+y^2-1}$,

   (b) $\log(y-x^2)$,

   (c) $\arcsin\dfrac{x}{y}$,

   (d) $\exp\left(\dfrac{-1}{x^2+y^2+z^2}\right)$.

   Here arcsin is the branch of the inverse of the sine function from $[-1,1]$ to $[-\pi/2, \pi/2]$.

   Solution.

   (a) The natural domain is where $x^2+y^2-1 \neq 0$, that is, $\{(x,y) : x^2+y^2 < 1\} \bigcup \{(x,y) : x^2+y^2 > 1\}$.

   (b) The log function is defined on $(0,\infty)$. Hence the natural domain is $\{(x,y) : y > x^2\}$.

   (c) The arcsine function is defined on $[-1.1]$. Therefore, the natural domain is $\{(x,y);\ -1 \leq x/y \leq 1\}$.

   (d) The natural domain is $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Note that this function can be extended to a continuous function in $\mathbb{R}^3$ after setting $f(0,0,0) = 0$.

10. Use Theorem 4.10 to determine whether the following sets are open or closed:

    (a) $S_1 = \{x \in \mathbb{R}^n : \; p(x) = 0 \}$ where p is a polynomial.

    (b) $S_2 = \{(x, y) \in \mathbb{R}^2 : \cos x^2 - \sin^3 xy \leq 1\}$.

    (c) $S_3 = \{(x, y, z) : \; x^2 + y^2 < \sin(x + z) < 28z^2\}$.

    Solution.

    (a) As a polynomial is continuous everywhere, $S_1 = p^{-1}(\{0\})$ is closed.

    (b) The function $f(x, y) \equiv \cos x^2 - \sin^3 xy$ is continuous in $\mathbb{R}^2$. S_2 is of the form $\{(x, y) : f(x, y) \leq 1\}$, hence it is closed.

    (c) The set $\{(x, y, z) : \; x^2 + y^2 < \sin(x + z)\}$ and the set $\{(x, y, z) : \; \sin(x + z) < 28z^2\}$ are two open sets. S_3 is the intersection of two open sets (see Problem 2), hence it is open.